N/DS Session 10 (complement)

$\boxed{3}$ i) Let (μ, v) be an eigenpair of B; i.e. $Bv = \mu v$

$B = A + 2I$

$Bv = (A + 2I)v = \mu v \iff Av = (\mu - 2)v$

$\iff (\mu - 2, v)$ is an eigenpair of A

Since $B = A + 2I$ has zero diagonal

$(Bv)_j = v_{j-1} + v_{j+1} = \mu v_j \qquad j = 1, \dots, N$

with $v_0 = v_{N+1} = 0$

ii) $v_{j-1} + v_{j+1} = \mu v_j$ is called a difference equation

A classical Ansatz for these types of equations is $v_j = C x^j$

where C and x are constants.

Plugging the Ansatz in the equation, we obtain

$$C x^{j-1}(1 - \mu x + x^2) = 0$$

If $C, x \neq 0$, x solves the quadratic equation

$$p(x) = x^2 - \mu x + 1 = 0$$

Denoting x_1 and x_2 the two roots $p(x) = (x - x_1)(x - x_2)$

By identification $\begin{cases} x_1 + x_2 = \mu \\ x_1 x_2 = 1 \end{cases}$

and the general solution of the difference equation is $v_j = c_1 x_1^j + c_2 x_2^j$

for $x_1 \neq x_2$

From the conditions $v_0 = v_{N+1} = 0$, we deduce that

$v_0 = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 = -c$

$$v_{N+1} = 0 \Rightarrow c(x_1^{N+1} - x_2^{N+1}) = 0 \Rightarrow \left(\frac{x_1}{x_2}\right)^{N+1} = 1 = e^{2\pi i k}$$

Thus $\quad x_1 = x_2 \, e^{\frac{2\pi i k}{N+1}} \quad k = 0, \dots, N \quad$ (but for $k = 0$, $x_1 = x_2$ does not meet the requirement)

Moreover $\quad x_1 x_2 = 1 \Rightarrow x_2^2 \, e^{\frac{2\pi k i}{N+1}} = 1$

$$\Rightarrow x_2 = e^{-\frac{\pi k i}{N+1}}$$

$$\Rightarrow x_1 = e^{\frac{\pi k i}{N+1}}$$

Thus, $\quad \mu^{(k)} = x_1 + x_2 = 2\,\mathrm{Re}\left(e^{\frac{\pi k i}{N+1}}\right) = 2\cos\left(\frac{\pi k}{N+1}\right) \quad k = 1, \dots, N$

Finally, the eigenvalues of A are $\mu^{(k)} - 2 = 2\cos\left(\frac{\pi k}{N+1}\right) - 2$

and the eigenvectors are $v_j^{(k)} = c(x_1^j - x_2^j)$

$$= c\left(e^{\frac{\pi k j i}{N+1}} - e^{-\frac{\pi k j i}{N+1}}\right)$$

$$= 2ic\,\mathrm{Im}\left(e^{\frac{\pi k j i}{N+1}}\right)$$

$$= \hat{c}\sin\left(\frac{\pi k j}{N+1}\right) \quad (\text{with } \hat{c} = 2ci)$$

$\boxed{4}$ i) Using Taylor expansions, we obtain

$$v(x_{i\pm1}) = v(x_i \pm \Delta x) = \sum_{k=0}^{\infty} (\pm 1)^k \frac{\Delta x^k}{k!} v^{(k)}(x_i)$$

Hence $\quad \dfrac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{\Delta x^2} = v^{(2)}(x_i) + \dfrac{\Delta x^2}{12} v^{(4)}(x_i) + O(\Delta x^3)$

ii) This expression holds provided v is at least $C^4(\Omega)$

iii) If v is only $C^2(\mathbb{R})$,

$$v(x_{i\pm 1}) = v(x_i) + (\pm 1)\Delta x\, v^{(1)}(x_i) + (\pm 1)^2 \frac{\Delta x^2}{2} v^{(2)}(x_i) + R_2^{\pm}(\Delta x)$$

where $R_2^{\pm}(\Delta x) = o(\Delta x^2)$ is the residual

$$\lim_{\Delta x \to 0} \frac{R_2^{\pm}(\Delta x)}{\Delta x^2} = 0$$

The approximation is still convergent but we cannot say anything about the order of convergence.